# 1.4. Transform-Domain Representation of Discrete Signals and LTI Systems

## 1.4.1. Z -Transform

**Definition:** The $Z$ – transform of a discrete-time signal *x(n)* is defined as the power series:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-k} \qquad\qquad X(z) = Z[x(n)]$$

where $z$ is a complex variable. The above given relations are sometimes called **the direct *Z* - transform** because they transform the time-domain signal *x(n)* into its complex-plane representation *X(z)*.

Since $Z$ – transform is an infinite power series, it exists only for those values of $z$ for which this series converges. The **region of convergence** of *X(z)* is the set of all values of $z$ for which *X(z)* attains a finite value.

The procedure for transforming from z – domain to the time-domain is called **the inverse Z – transform**. It can be shown that the inverse *Z* – transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad x(n) = Z^{-1}\left[ X(z) \right]$$

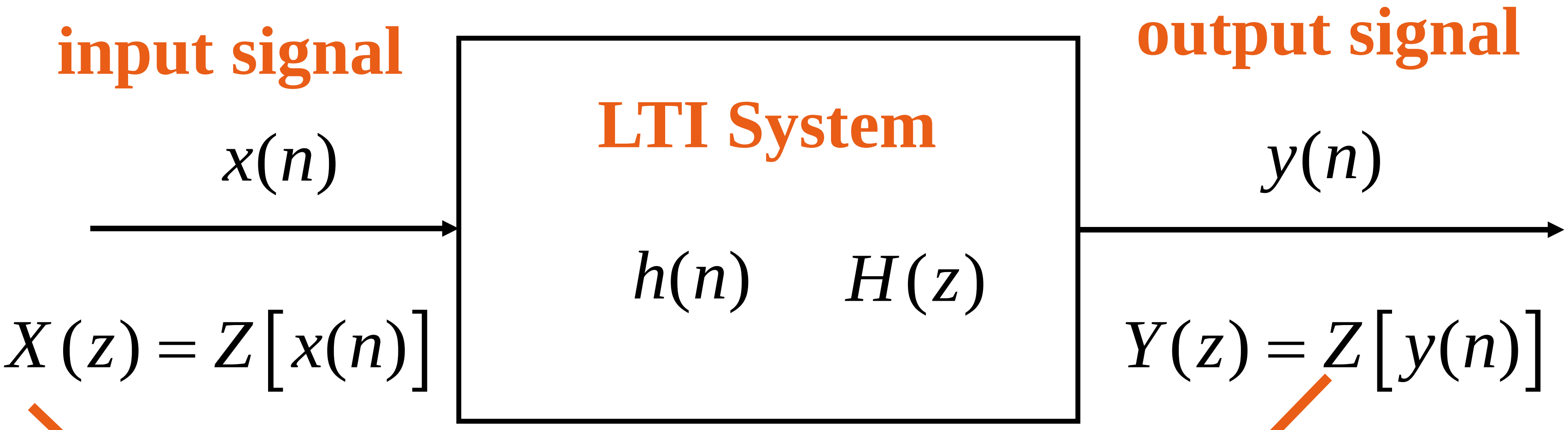
where *C* denotes the closed contour in the region of convergence of *X(z)* that encircles the origin.

## 1.4.2. Transfer Function

The LTI system can be described by means of **a constant coefficient linear difference equation** as follows

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

Application of the *Z*-transform to this equation under zero initial conditions leads to the notion of **a transfer function**.




$$H(z) = Z[h(n)]$$

**Transfer function:** the ratio of the $Z$ - transform of the output signal and the $Z$ - transform of the input signal of the LTI system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Z[y(n)]}{Z[x(n)]}$$

LTI system: the *Z*-transform of the constant coefficient linear difference equation under zero initial conditions:

$$y(n)=\sum_{k=0}^{N} b(k)x(n-k)-\sum_{k=1}^{M} a(k)y(n-k)$$

$$Y(z)=\sum_{k=0}^{N} b(k)z^{-k}X(z)-\sum_{k=1}^{M} a(k)z^{-k}Y(z)$$

The transfer function of the LTI system:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}$$

*H(z)*: may be viewed as a rational function of a complex variable $z$ ($z^{-1}$).

# 1.4.3. Poles, Zeros, Pole-Zero Plot

Let us assume that *H(z)* has been expressed in its irreducible or so-called factorized form:

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{N} (z - z_k)}{\prod_{k=1}^{M} (z - p_k)}$$

**Zeros of *H(z)*:** the set $\{z_k\}$ of *z*-plane for which $H(z_k)=0$

**Poles of *H(z)*:** the set $\{p_k\}$ of *z* -plane for which $H(p_k) \to \infty$

**Pole-zero plot:** the plot of **the zeros** and **the poles** of *H(z)* in the z-plane represents a strong tool for LTI system description.

**Example:** the 4-th order Butterworth low-pass filter, cut off frequency $\omega_1 = \pi/3$ .

$b$ =[ 0.0186    0.0743    0.1114    0.0743    0.0186 ]

$a$ =[ 1.0000   -1.5704   1.2756   -0.4844    0.0762 ]

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

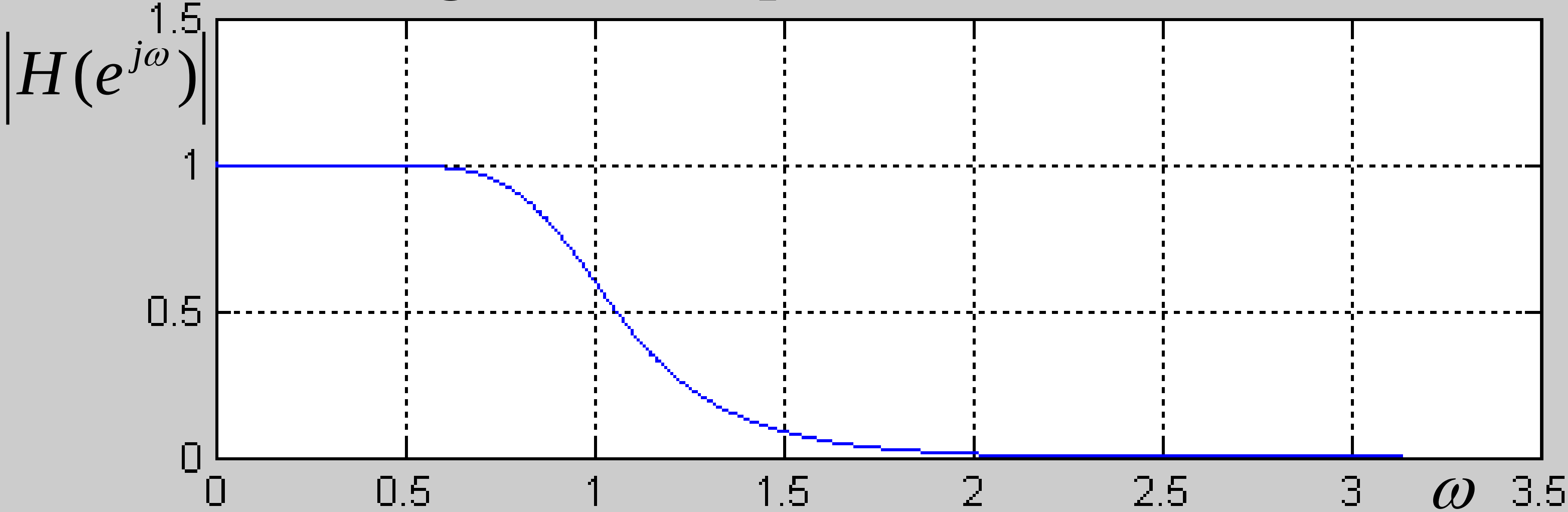
$z_1$= -1.0002, $z_2$= -1.0000 + 0.0002j

$z_3$= -1.0000 - 0.0002j, $z_4$= -0.9998

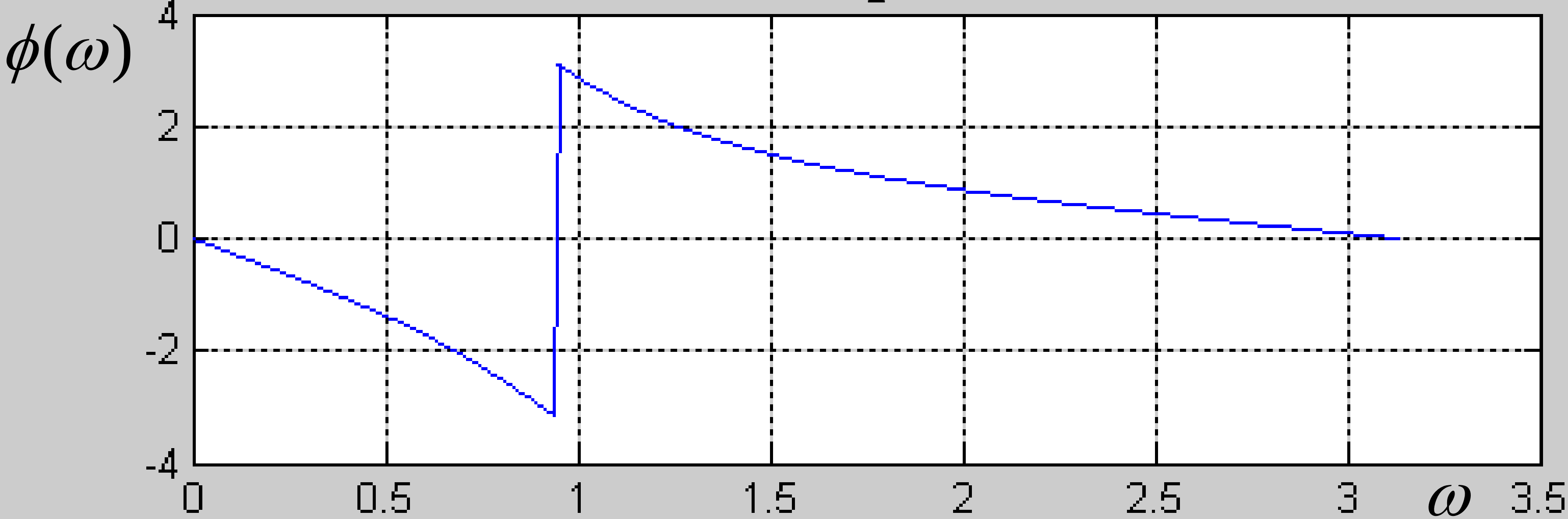
$p_1$= 0.4488 + 0.5707j, $p_2$= 0.4488 - 0.5707j

$p_3$= 0.3364 + 0.1772j, $p_4$= 0.3364 - 0.1772j

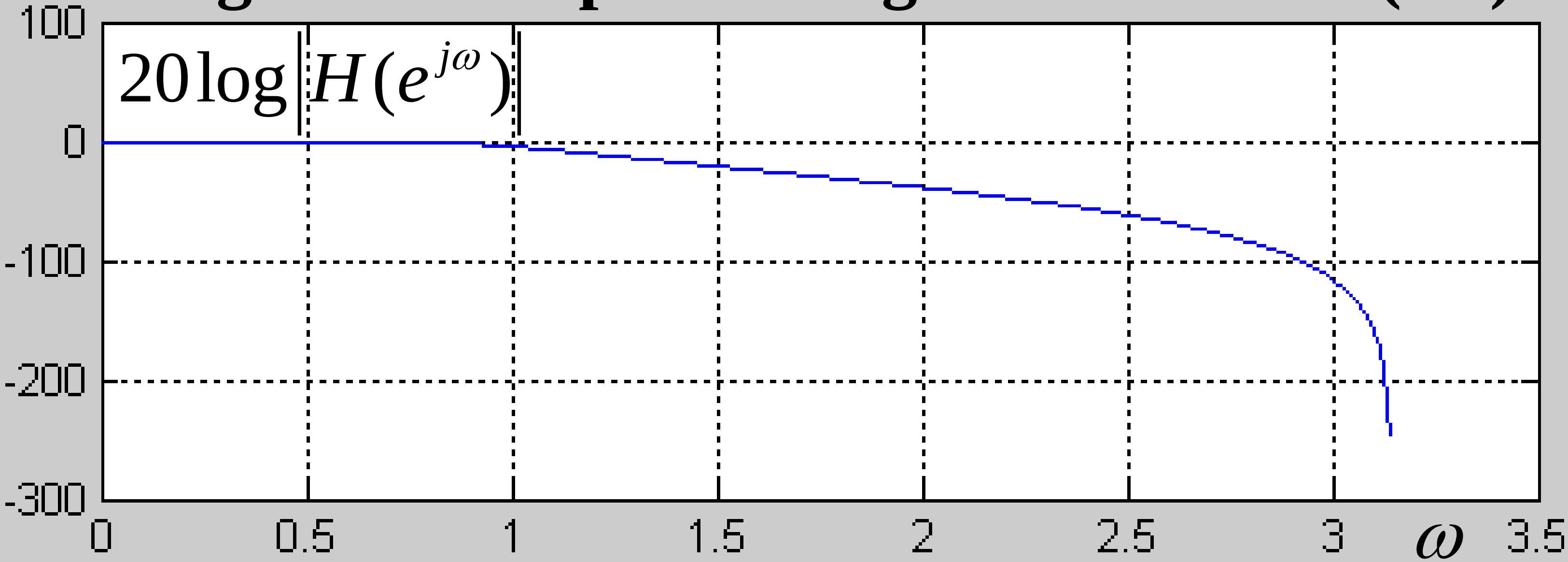
# Magnitude Response: Linear Scale



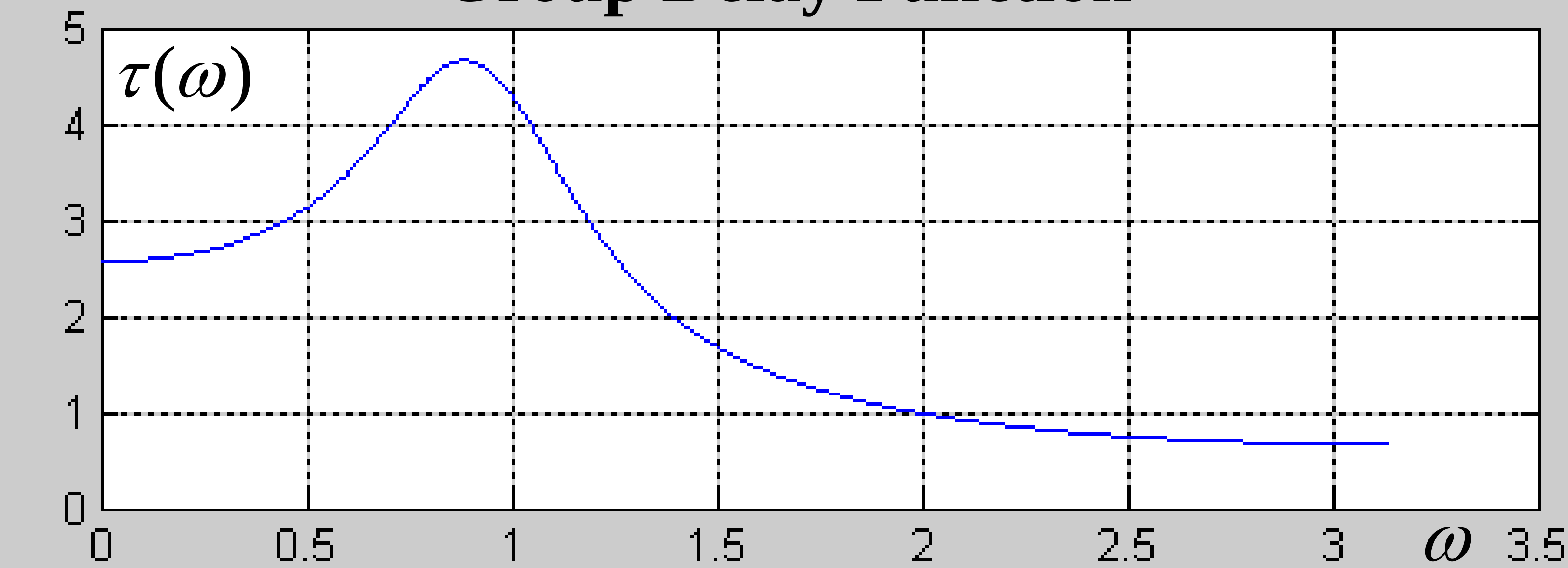

# Phase Response



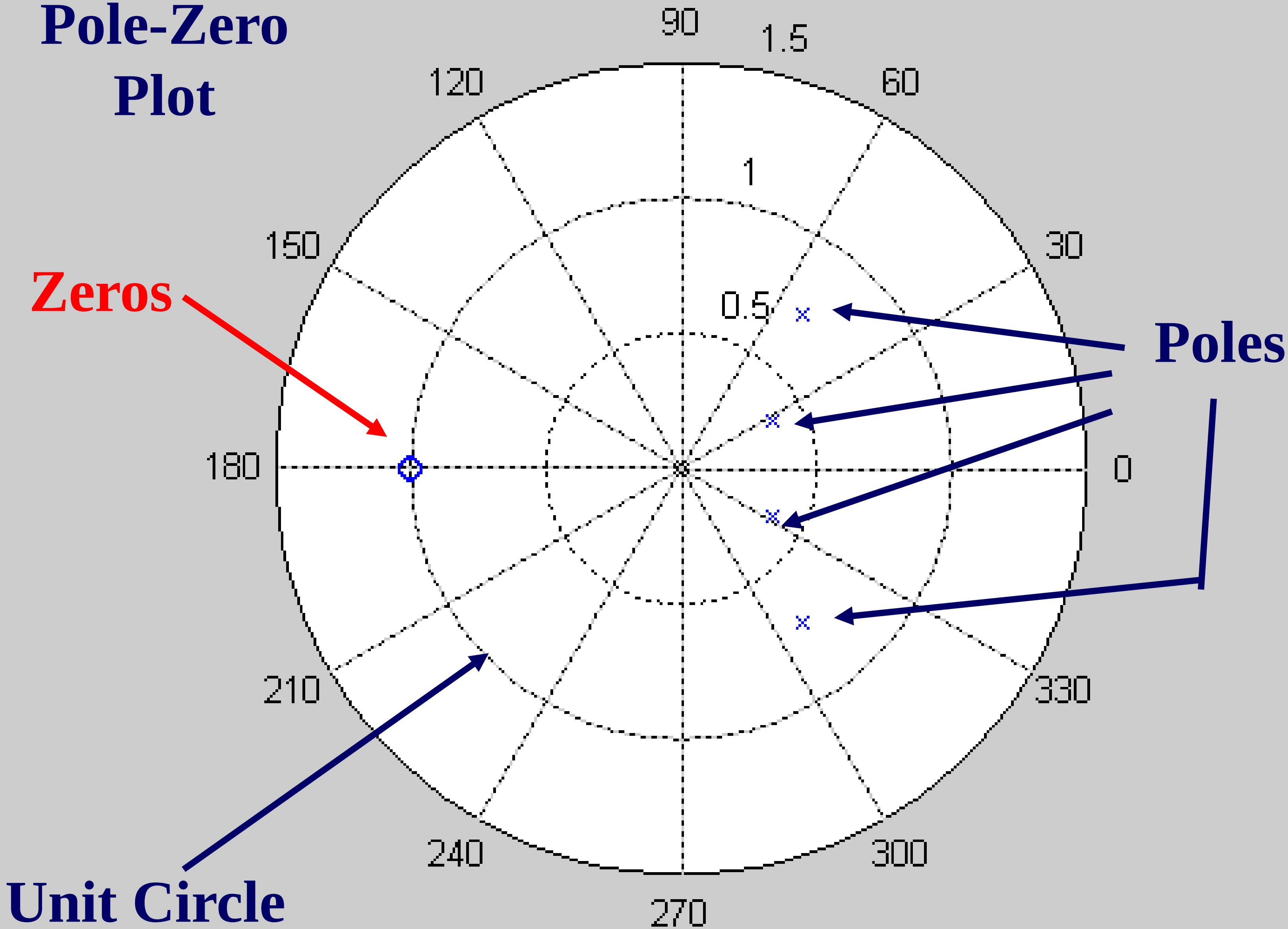
# Magnitude Response: Logarithmic Scale (dB)



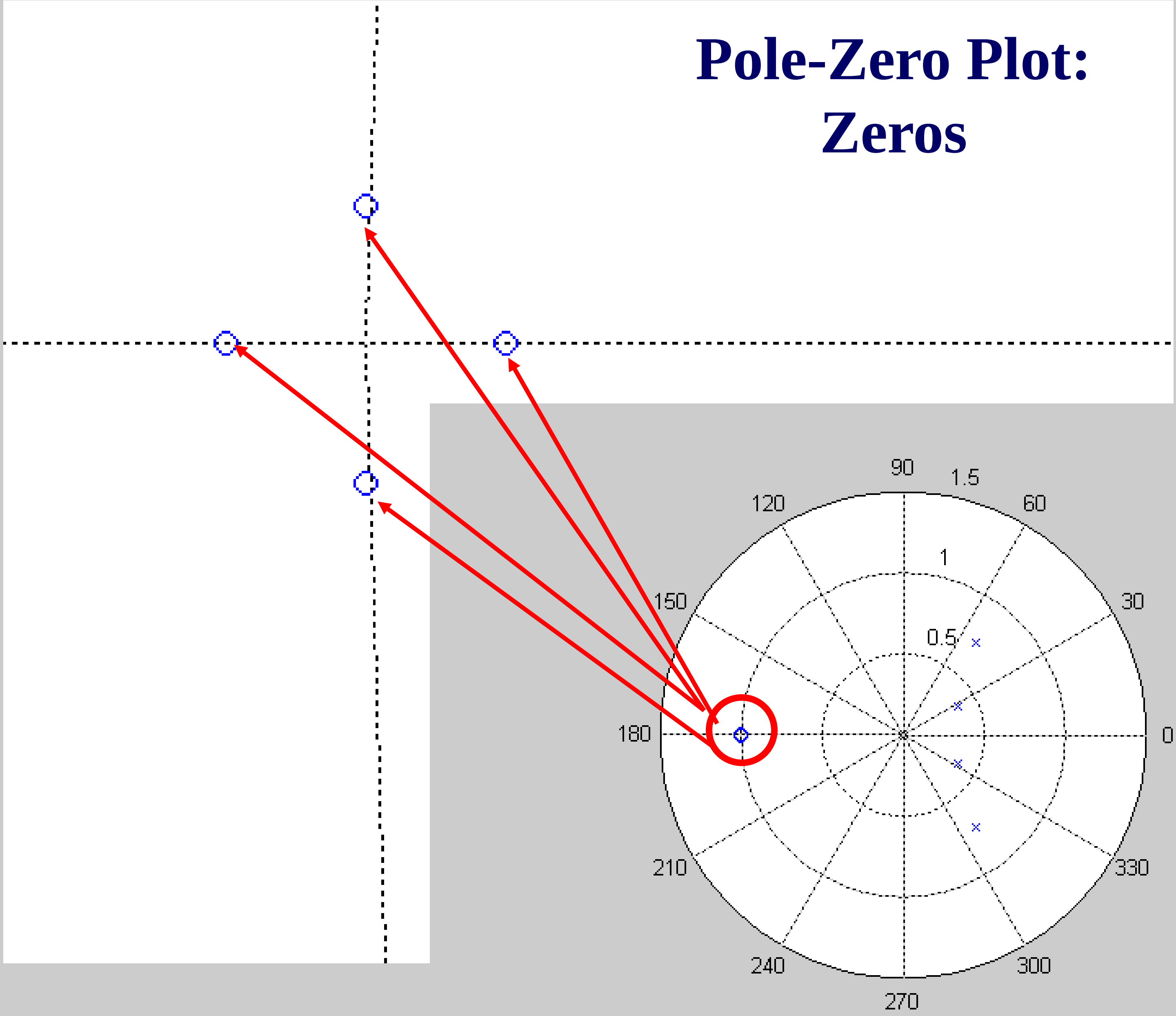

# Group Delay Function



# Pole-Zero Plot



# Pole-Zero Plot: Zeros



## 1.4.4. Transfer Function and Stability of LTI Systems

**Condition:** **LTI system is BIBO stable if and only if** the unit circle falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LTI system, the stability condition is equivalent to the requirement that **the transfer function *H(z)* has all of its poles inside the unit circle.**

## Example 1: stable system

$$H(z) = \frac{1 - 0.9z^{-1} + 0.18z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

$z_1 = 0.3 \quad p_1 = 0.4000 + 0.6928j \quad |p_1| = 0.8 < 1$

$z_2 = 0.6 \quad p_2 = 0.4000 - 0.6928j \quad |p_2| = 0.8 < 1$

## Example 2: unstable system

$$H(z) = \frac{1 - 0.16z^{-2}}{1 - 1.1z^{-1} + 1.21z^{-2}}$$

$z_1 = 0.4 \quad p_1 = 0.5500 + 0.9526j \quad |p_1| = 1.1 > 1$

$z_2 = -0.4 \quad p_2 = 0.5500 - 0.9526j \quad |p_2| = 1.1 > 1$

# 1.4.5. LTI System Description. Summary

## Time – Domain:

constant coefficient linear difference equation

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$h(n)$

## Z – Domain:

transfer function

Z

$Z^{-1}$

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}$$

## Frequency – Domain:

FT frequency response

$FT^{-1}$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{N} b(k)e^{-j\omega k}}{1+\sum_{k=1}^{M} a(k)e^{-j\omega k}}$$

$z = e^{j\omega}$ $e^{j\omega} = z$

**Time – Domain:** impulse response $h(k)$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

**$Z$ – Domain:** transfer function $H(z)$

$$H\left(e^{j\omega}\right) = H\left(z\right)_{z=e^{jw}} \qquad h(n) = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz$$

**Frequency – Domain:** frequency response $H\left(e^{j\omega}\right)$

$$H(z) = H\left(e^{j\omega}\right)_{e^{j\omega}=z} \qquad h(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega k}d\omega$$